Inscribing a circle in a hypercube

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Problem. What is the radius of the largest (2-dimensional) circle that can be inscribed in n-dimensional hypercube whose edges have unit length?

Without loss of generality we assume that the hypercube is centered at the origin and the coordinates of its vertices are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$. Then the set of points H_n that belong to the hypercube is

$$H_n = \left\{(x_1, x_2, \ldots, x_n) \mid -\tfrac{1}{2} \leq x_i \leq \tfrac{1}{2}\right\}. \tag{1}$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and assume that

$$\|\mathbf{x}\| = \|\mathbf{y}\| = r \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = 0. \tag{2}$$

A circle $C_{\mathbf{x},\mathbf{y}}$ of radius r that is centered at the origin and belongs to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ is the following set of points:

$$C_{\mathbf{x},\mathbf{y}} = \{\mathbf{x}\cos\alpha + \mathbf{y}\sin\alpha \mid 0 \leq \alpha < 2\pi\}. \tag{3}$$

The condition $C_{\mathbf{x},\mathbf{y}} \subset H_n$ is equivalent to

$$\forall \alpha \in \mathbb{R}, \forall i \in \{1, \ldots, n\} : \ |x_i \cos\alpha + y_i \sin\alpha| \leq \tfrac{1}{2}. \tag{4}$$

Note that

$$\max_{\alpha \in \mathbb{R}} |x\cos\alpha + y\sin\alpha| = \sqrt{x^2 + y^2}, \tag{5}$$

since $x\cos\alpha + y\sin\alpha$ is the first component of

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{6}$$

which is just $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ rotated clockwise by angle α. Thus we can eliminate α from condition (4) and rewrite it as follows:

$$\forall i \in \{1, \ldots, n\} : \ x_i^2 + y_i^2 \leq \tfrac{1}{4}. \tag{7}$$

Now we can state the problem of finding the largest circle inscribed in the hypercube as the following optimization problem:

$$\text{maximize } r \text{ subject to : } \quad \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = r^2, \tag{8}$$

$$\sum_{i=1}^{n} x_i y_i = 0, \tag{9}$$

$$x_i^2 + y_i^2 \leq \tfrac{1}{4}, \ \forall i \in \{1, \ldots, n\}. \tag{10}$$

From (8) and (10) we get

$$\sum_{i=1}^{n} (x_i^2 + y_i^2) = 2r^2 \leq \frac{n}{4}, \tag{11}$$

which gives us an upper bound on r:

$$r \leq \sqrt{\frac{n}{8}}. \tag{12}$$

It remains to show that this upper bound can be achieved for all $n \geq 2$.

To achieve the upper bound, all inequalities in (10) must be saturated. Thus the optimality conditions read:

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2, \quad \sum_{i=1}^{n} x_i y_i = 0, \quad x_i^2 + y_i^2 = \tfrac{1}{4}, \ \forall i \in \{1, \ldots, n\}. \tag{13}$$

When $n \geq 2$ is even, we can use the following two vectors to satisfy (13):

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \cdots & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}. \tag{14}$$

One can also directly see that $C_{\mathbf{x},\mathbf{y}} \subset H_n$ holds in this case.

When $n \geq 3$ is odd, we use

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \cdots & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}. \tag{15}$$

It is interesting and counter-intuitive that r can get arbitrarily large, when n increases. In particular, $r > 1$ when $n > 8$.

Note: We have not justified the assumption that the largest circle is centered at the origin. This assumption seems reasonable. However, one still has to rule out the possibility that a larger circle, not centered at the origin, can be found.